Scaling properties of a one-dimensional sandpile model with grain dissipation

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We have studied a stochastic sandpile model with grain dissipation as a generalization of the Oslo sandpile model. During a toppling event, grains are removed from the pile with a probability p. Scaling arguments and simulations suggest that an arbitrarily small dissipation rate p yields a noncritical behavior, in contrast to the robust critical behavior of the Oslo sandpile model.

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I. INTRODUCTION

Sandpile models have been intensively studied since their introduction as generic examples of self-organized criticality (SOC) by Bak, Tang, and Wiesenfeld [1,2]. SOC was introduced to address the behavior of a class of slowly driven systems that showed scale invariant behavior in space and time without the need for extrinsic tuning of the system parameters. Self-organized criticality has been suggested as a common mechanism for these systems, and has been applied to phenomena spanning from earthquakes to economics [3–6].

SOC provides a useful way of categorizing systems and suggests that the methods used to analyze phase transitions in statistical mechanics may be used to analyze SOC systems. These include finite size scaling and the estimation of power law exponents for different distributions. Power law exponents may be used to categorize systems into universality classes.

Given the extensive theoretical and numerical studies of SOC there has been surprisingly little experimental activity. However, slowly driven granular piles have been studied by several experimental groups [7–11]. The question of whether granular piles may be described as self-organized critical has only recently been resolved. An experiment on a pile of rice showed a power law distribution of energy dissipation in avalanches in a pile of long-grained rice. Piles of rounder rice produced a stretched exponential distribution [12].

A stochastic cellular automaton model was introduced by Frette [13] to mimic the rearrangement disorder in the rice pile. The model was reanalyzed by Christensen *et al.* [14] and has later been referred to as the Oslo sandpile model (OSM). This model gave a power law for the energy dissipation distribution, but the exponent did not match the experimental value and the model contained no parameter that could be tuned to show a continuous transition from a critical to a noncritical behavior. It has later been shown that the OSM is a discrete version of a quenched Edwards-Wilkinson equation [15,16].

Markošová *et al.* [17] studied a simple extension of a model introduced by Amaral and Lauritsen [18,19] where they could relate the critical behavior to intermediate values of a toppling probability. A mean field approach to this model has been studied by Slanina [20]. Sandpile models with dissipation of slope have been studied by Manna *et al.* [21].

In this paper, we introduce a generalization of OSM, which allows for grain dissipation from every position in the pile, and study its dynamics by simulations. In this model, slope is conserved, but mass is not, so it is different from both the model by Markošová *et al.* [17] and Manna *et al.* [21]. The simulation data showed linear system size dependence for the scaling of small avalanches as in the rice pile experiment. The introduction of grain dissipation gave a cutoff in the probability density for avalanche sizes as a function of dissipation rate. The cutoff was present even for arbitrarily small dissipation rates. We found that the probability density of dissipated energy as a function of energy, system size, and grain dissipation rate behaved as a generalized homogeneous function.

II. SIMULATIONS

The Oslo sandpile model consists of a row of sites where each site is assigned an integer height h_i. One unit height corresponds to one grain. Grains enter the system at $i=1$ and leave the system at $i=L$, where L is the system size. A critical slope z_i^c is assigned to each site i. The critical slopes are chosen randomly to be either 1 or 2. A grain at the ith site moves one site toward the outlet when the slope $z_i=h_i-h_{i+1}$ exceeds the critical slope, and a new random value for the critical slope is drawn. As a consequence of grain movement, the slopes are updated according to the following rule:

$$\text{If } i=1: \begin{cases} z_1 & \to z_1-2 \\ z_2 & \to z_2+1. \end{cases}$$

$$\text{If } 1<i<L: \begin{cases} z_i & \to z_i-2 \\ z_{i\pm1} & \to z_{i\pm1}+1. \end{cases}$$

$$\text{If } i=L: \begin{cases} z_L & \to z_L-1 \\ z_{L-1} & \to z_{L-1}+1. \end{cases} \tag{1}$$

The slopes are updated until all sites are stable, that is, $z_i \leqslant z_i^c$ for all i. Then another grain is added at position $i=1$: $z_1 \to z_1+1$. An avalanche is defined as the series of updates that follow from the addition of one grain.

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We introduce a finite probability p for a grain to leave the pile when it topples. This is governed by the following dissipation rule:

$$z_i \to z_i - 1$$

$$z_{i-1} \to z_{i-1} + 1, \tag{2}$$

which can be interpreted as an inertial effect, that is, that a grain simply bounces off the pile. We note that the model conserves slope. The model with $p=0$ gives the Oslo sandpile model, whereas the model with $p=1$ will remove the added grain without changing the state of the model. The model with $p=1$ is similar to the boundary driven one-dimensional Bak-Tang-Wiesenfeld in this respect, since the one-dimensional Bak-Tang-Wiesenfeld model also retains its state before and after an avalanche when it is in its stationary state. The dissipated energy will be proportional to the system size for both models, but the models do differ in their avalanche dynamics.

Avalanches are characterized by the potential energy dissipated during an event. One unit of energy is defined as the loss in potential energy when one grain moves down one height unit. In the following the energy ϵ will be normalized by the above defined unit of energy, so ϵ becomes nondimensional.

The measure used to characterize the statistical behavior of the system is the energy probability density f_E, where $f_E(\epsilon)d\epsilon$ is defined as the probability for having an avalanche of size with energy between ϵ and $\epsilon+d\epsilon$.

The energy probability density for the OSM simulations was reported [14] to behave like a power law $f_E(\epsilon) \propto \epsilon^{-\alpha}$ except for small avalanches and finite size effects. f_E is also assumed to be a homogeneous function of the variables L and ϵ, which gives the scaling relations

$$f_E(\epsilon, L) = L^{-\beta} g(\epsilon L^{-\nu}) = \epsilon^{-\beta/\nu} \tilde{g}(L\epsilon^{-1/\nu}), \tag{3}$$

where g and $\tilde{g}$ are scaling functions. The relation $\alpha=\beta/\nu$ follows directly from Eq. (3). The finite size scaling exponents were estimated to be $\beta \approx 3.5$, $\nu \approx 2.25$, and $\alpha \approx 1.55$ [14]. However, the scaling assumptions are only valid for $L, \epsilon \gg 1$. L introduces a finite size cutoff in the energy $\epsilon_L \propto L^{\nu}$. The average amount of dissipated energy $\langle \epsilon \rangle$ is proportional to the average number of topplings $\langle s \rangle$. It was shown [15,22] that $\langle s \rangle \propto L$ giving the scaling relation $2\nu - \beta = 1$.

Grains that are dissipated cause a discontinuity in the probability density which scales with the system size, adding unwanted complexity to the interpretation of simulation data. The data become more manageable by disregarding the energy loss from dissipated grains. In the following analysis, we have disregarded these grain dissipating events when calculating the energy dissipated in avalanches. Figure 1 shows the probability density for energy dissipated in avalanches. For $p>0$ we observe a cutoff which falls short of the finite size cutoff for the Oslo sandpile model.

Figure 2 shows the energy probability densities for systems with $p=0.01$ and varying L value. The graphs for $L=400$ to $L=3200$ have a cutoff which is independent of the system size, thus the dissipation mechanism introduces an effective cutoff energy ϵ_p, which becomes visible in larger systems. The slope fitted to the linear part of the log-log plot gives $\alpha = 1.56 \pm 0.07$, indicating that the model behaves similar to the OSM for $\epsilon < \epsilon_p$.

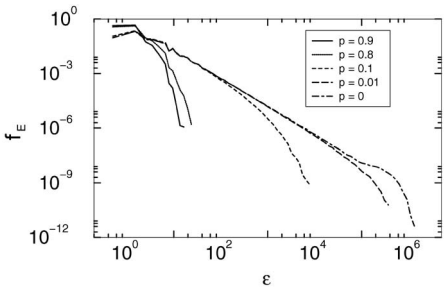

FIG. 1. Log-log plot of the probability density $f_E(\epsilon, L)$ as a function of energy dissipated ϵ by grains following rule (1) only. The system size is $L=400$. The grain dissipation probability p varied from 0 to 0.9. The data were binned, and the bin sizes were increased by a factor of 1.2.

We argue that a change from the OSM universality class is reasonable for nonzero p, by showing that scaling ansatz (3) cannot hold, given that the fraction of grains following the dissipation rule is independent of the avalanche size. From scaling ansatz (3) it follows that the size of the largest avalanche scales as $\epsilon^* \propto L^{\nu}$. The number of topplings N is proportional to ϵ^*, since the slope values are bounded and the energy dissipated in a toppling at site i is $z_i - 1$. For nonzero p there is a finite probability that a toppling does not contribute to the energy. If M is the total number of topplings in the nonconservative model, then $(1-p)M$ is the number of energy contributing topplings N. The number of grains leaving the pile is $pM = p/(1-p)N$, for later use we define $\mu = p/(1-p)$, which is related to the system size by $\mu N \propto \mu\epsilon \propto \mu L^{2.25}$. The number of grains in the pile scales as L^2 and this is incompatible with the number of grains that are dissipated. Thus scaling ansatz (3) cannot be valid for large system sizes.

Including the dissipation parameter μ in the generalized homogeneous form of the probability density leads to a scaling ansatz

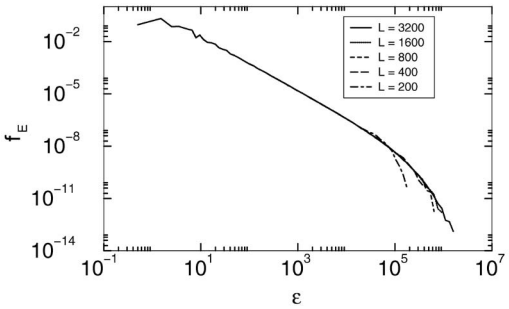

FIG. 2. A log-log plot of the energy probability density $f_E(\epsilon, L, p)$ as a function of dissipated energy ϵ. $p=0.01$ and the system sizes vary from 200 to 3200. The graphs show a cutoff which is independent of system size for $L>400$. The data were binned, and the bin sizes were increased by a factor of 1.2.

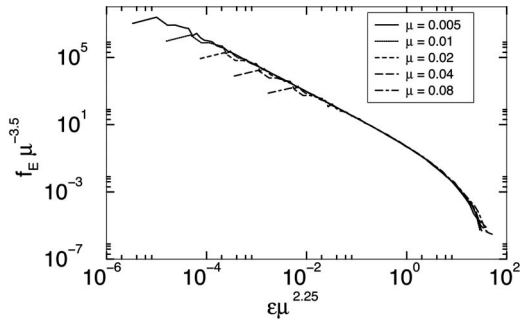

FIG. 3. A log-log plot of the energy probability density f_E as a function of dissipated energy ϵ. The system size used for $\mu=0.005$ was $L=3200$ and the rest of the system size were calculated from the scaling relation $\mu \propto L^{-\omega}$ with $\omega=1$. The x and y axes were rescaled by respectively $\mu^{\nu/\omega}$ and $\mu^{-\beta/\omega}$. A good data collapse was given by $1 \leqslant \omega \leqslant 1.1$. The data was binned, and the bin size increased by a factor of 1.2.

$$f_E(\epsilon, L, \mu) = \lambda f_E(\epsilon\lambda^a, L\lambda^b, \mu\lambda^c), \tag{4}$$

where λ is an arbitrary parameter, $a=\nu/\beta$, $b=1/\beta$, and $c=-\omega/\beta$. β and ν are the finite size scaling exponents from Eq. (3), and ω is a scaling exponent related to the cutoff due to grain dissipation.

The following expressions are obtained by choosing λ so that it cancels the L, ϵ, or μ dependence in f_E

$$f_E(\epsilon, L, \mu) = \begin{cases} \epsilon^{-\beta/\nu} h(L\epsilon^{-1/\nu}, \mu\epsilon^{\omega/\nu}) \\ L^{-\beta} \tilde{h}(\epsilon L^{-\nu}, \mu L^{\omega}) \\ \mu^{\beta/\omega} \tilde{\tilde{h}}(\epsilon\mu^{\nu/\omega}, L\mu^{1/\omega}), \end{cases} \tag{5}$$

where h, $\tilde{h}$, and $\tilde{\tilde{h}}$ are scaling functions. A data collapse of f_E by rescaling with μ according to Eq. (5) and $\omega=1$, is shown in Fig. 3. The system sizes and μ values ranged from $L=3200$, $\mu=0.005$ to $L=200$, $\mu=0.08$. A data collapse was achieved for $1.0 \leqslant \omega \leqslant 1.1$.

We can relate the ω exponent to the scaling of the average dissipated energy for grain conserving topplings

$$\langle \epsilon \rangle_L = \int \epsilon f_E(\epsilon, L, \mu) d\epsilon = \int \epsilon \mu^{\beta/\omega} \tilde{\tilde{h}}(\epsilon\mu^{\nu/\omega}, L\mu^{1/\omega}) d\epsilon$$

$$= \mu^{(\beta-2\nu)/\omega} \int u \tilde{\tilde{h}}(u, L\mu^{1/\omega}) du \underset{L\to\infty}{\propto} \mu^{-1/\omega}, \tag{6}$$

where we have used the scaling relation $2\nu-\beta=1$, and the integral converges since dissipation introduces a cutoff in the energy.

We can show that $\omega=1$ by calculating the average number of topplings. Let δz_i be the number of units of slope added to a site during a run of K grain addings. If we assume that the sandpile is in a stationary state, slope that is added to a site that must be transported off the site. From toppling rules (1) and (2) we have that site i will reduce its slope by two with probability $1-p$ and by one with probability p, thus the number of times site i topples is approximately $\delta z_i/(2-p)$ except for $i=L$ where the site must topple δz_i times. In the Appendix we show that

$$\delta z_i = \begin{cases} K(2-p)(1-p)^{i-1}, & \text{for } i < L \\ K(1-p)^{L-1}, & \text{for } i = L, \end{cases} \tag{7}$$

and the average number of topplings can be calculated

$$\langle s \rangle_L = \frac{1}{K}\left(\sum_{i=1}^{L-1} \frac{\delta z_i}{2-p} + \delta z_i \right) = \frac{1}{p}\{1 + O[(1-p)^{L-1}]\}. \tag{8}$$

For $L\to\infty$ and $\mu \ll 1$ we have that $\langle s \rangle \approx 1/\mu$, by comparison with Eq. (6) we obtain that $\omega=1$.

We can now reanalyze the effect of the cutoff due to dissipation by using scaling ansatz (5). Equation (5) can be written as

$$f_E(\epsilon, L, \mu) = \epsilon^{-\beta/\nu} f_E(\epsilon_0, L\epsilon^{-1/\nu}, \mu\epsilon^{\omega/\nu}), \tag{9}$$

where ϵ_0 is a constant and where we recognize $f_E(\epsilon_0, x, y)$ as $h(x,y)$. If $y\to\infty$ then $h(x,y)\to 0$ independent of x, since this is equivalent to $f_E(\epsilon_0, x, y)\to 0$, as $y\to\infty$. This can be interpreted as sending μ to infinity in the model, which implies that all grains are dissipated as soon as they enter the pile. The scaling ansatz gives a cutoff in the energy due to dissipation, as $\mu^{-\nu/\omega}$, since $y=\mu\epsilon^{\omega/\nu}$. This is the effective cutoff observed in Figs. 2 and 3.

III. CONCLUSIONS

In this paper, we have studied a basic dissipative cellular automaton, aiming to illustrate the effects of dissipation in the rice pile experiment, in order to capture some of the qualitative behavior observed in the experiment.

The introduction of dissipation in OSM changes the behavior radically. A cutoff in the energy appears, which is solely caused by the rate of dissipation. This is in accordance with the findings of Vespignani *et al.* [23] and Vespignani and Zapperi [24]. We were able to include the dissipation parameter in the probability density while keeping its generalized homogeneous form, so dissipation introduced a new scaling exponent.

The simulation results suggest that the change in behavior observed in the experiment when the type of rice is changed from a long-grained rice to a rounder rice, may be interpreted as a dissipation effect. Assuming that a pile of long-grained rice will build a rougher surface which will slow down the grains [12].

For the long-grained rice that displayed SOC, the cutoff energy due to dissipation would be large compared to the energies reached in the experiment. This may explain why the distribution was not affected by dissipation effects.

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APPENDIX: DERIVATION OF δz_i

Here, we derive the expression for δz_i. From toppling rules (1) and (2) we get the interaction between sites and by

using the number of topplings for a site i is $\delta z_i/(2-p)$ if $i<L$, and δz_i if $i=L$, we derive the recursion relations for δz_i for a run of K avalanches

$$\delta z_1 = \frac{\delta z_2}{(2-p)} + K, \tag{A1}$$

$$\delta z_{1<i<L-1} = \frac{\delta z_{i+1}}{(2-p)} + \frac{1-p}{2-p}\delta z_{i-1}, \tag{A2}$$

$$\delta z_{L-1} = \delta z_L + \frac{1-p}{2-p}\delta z_{L-2}, \tag{A3}$$

$$\delta z_L = \frac{1-p}{2-p}\delta z_{L-1}. \tag{A4}$$

By inserting Eq. (A4) for δz_L in Eq. (A3) we get

$$\delta z_{L-1} = (1-p)\delta z_{L-2}. \tag{A5}$$

We recognize Eq. (A2) as a second-order difference equation, with the characteristic equation

$$r^2 + (p-2)r + 1 - p = 0, \tag{A6}$$

with the solutions $r_1=1$ and $r_2=1-p$.

Equations (A1) and (A5) supply the boundary conditions for Eq. (A2), and with $p>0$ we obtain

$$\delta z_i = K(2-p)(1-p)^{i-1} \tag{A7}$$

for $1 \leqslant i \leqslant L-1$. δz_L is found by inserting the expression for δz_{L-1} in Eq. (A4)

$$\delta z_L = K(1-p)^{L-1}. \tag{A8}$$

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